

THE ACTION OF AN ALGEBRAIC TORUS ON THE AFFINE PLANE

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1. **Introduction.** Let G be a connected algebraic group, V a variety. G is said to operate regularly on V if we are given an everywhere defined rational map $g \times v \rightarrow g(v)$ of $G \times V \rightarrow V$ such that

- (1) $g_1(g_2(v)) = g_1g_2(v)$ for any $g_1, g_2 \in G, v \in V$,
- (2) $e(v) = v$ for any $v \in V$.

Denote by k an algebraically closed field, by A^2 the affine plane over k , by G_m the multiplicative group of the universal domain.

Our purpose in this paper is to study by elementary means the regular operation of G_m on A^2 . We shall denote by σ a regular operation of G_m on A^2 which is not the identity on A^2 , $\sigma: G_m \times A^2 \rightarrow A^2$, and by $\sigma(t)$ the restriction of this map given by $\sigma(t): t \times A^2 \rightarrow A^2$, where $t \in G_m$.

We recall that an algebraic torus is the direct product of a finite number of multiplicative groups.

2. **Change of coordinates in A^2 .** Let (x, y) be a system of coordinates for A^2 . Then if

$$(2.1) \quad \begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases} \quad \text{and} \quad \begin{cases} x = f'(x', y'), \\ y = g'(x', y'), \end{cases}$$

where $f, g \in k[x, y]$, $f', g' \in k[x', y']$, the system (x', y') will also be an allowable system of coordinates for A^2 . (Considered in one coordinate system the map $(x, y) \rightarrow (x', y')$ is an entire Cremona-transformation.)

3. **Semi-invariant polynomials.** Let a regular operation of G_m on A^2 be given by $\sigma(t): (t, x, y) \rightarrow (x^*, y^*)$ where $t \in G_m$, (x, y) is a coordinate system for A^2 , and $x^*, y^* \in k[x, y, t, t^{-1}]$.

If $f \in k[x, y]$; we define $\lambda_t f$, the t -translate of f , by

$$(3.1) \quad \lambda_t f(x, y) = f(x^*, y^*).$$

We say that f is a semi-invariant polynomial (abbreviated in the sequel as s.i.p.) of weight s if $\lambda_t f = t^s f$, where s is an integer. If $s = 0$, f is said to be an invariant polynomial.

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LEMMA 1. Let $f \in k[x, y]$ and suppose that $\lambda_t f = pf$, with $p \in k[x, y, t, t^{-1}]$. Then $p = t^s$, s an integer.

Proof. Indeed $\lambda_{t^{-1}}(\lambda_t f) = f$. Therefore $p(x, y, t^{-1}, t) \lambda_{t^{-1}} p(x, y, t, t^{-1}) = 1$, so that p is independent of x and y , and the result follows.

LEMMA 2. (a) Any factor of an s.i.p. is an s.i.p.

(b) Every semi-invariant rational function is the quotient of two s.i.p.

Proof. (a) Let $f_1, f_2, \dots, f_n$ be the irreducible factors of a semi-invariant polynomial; then $\lambda_t(f_1 \cdots f_n) = \lambda_t f_1 \lambda_t f_2 \cdots \lambda_t f_n = t^s f_1 \cdots f_n$.

Therefore

$$(3.2) \quad \lambda_t f_i = p(t, t^{-1}) \prod_{j \in I} f_j$$

where I is a subset of $1, 2, \dots, n$, $p(t, t^{-1}) \in k[t, t^{-1}]$. Taking $t = 1$ in (3.2) it follows that $\lambda_t f_i = p(t, t^{-1}) f_i$.

(b) Let $\lambda_t(f/g) = t^a f/g$; then $(\lambda_t f)g = t^a f(\lambda_t g)$. Since f and g are relatively prime, the result follows by a standard divisibility argument.

COROLLARY A. All polynomials of $k[x, y]$ which are semi-invariant under σ are products of irreducible s.i.p.

PROPOSITION 1. Every polynomial of $k[x, y]$ is a sum of s.i.p.

Proof. Let $f \in k[x, y]$, since $\lambda_t f \in k[x, y, t, t^{-1}]$ we have

$$(3.3) \quad \lambda_t f = t^a (f_0 + t f_1 + \dots + t^n f_n) = t^a \sum_{i=0}^n t^i f_i$$

where, for all i , $f_i \in k[x, y]$ and $n \geq 0$.

Since σ is a regular operation of G_m on A^2 , the following conditions are satisfied:

$$(3.4) \quad \begin{aligned} (1) \quad & \sigma(1) \text{ is the identity on } A^2, \\ (2) \quad & \sigma(tt_1) = \sigma(t)\sigma(t_1). \end{aligned}$$

From (3.4) it follows that

$$(3.5) \quad (tt_1)^a \sum_{i=0}^n t^i t_1^i f_i = t_1^a \sum_{i=0}^n t_1^i (\lambda_{t_1} f_i).$$

Identifying corresponding coefficients of t_1^{i+a} in the two members of (3.5) we get

$$(3.6) \quad \lambda_t f_i = t^{a+i} f_i.$$

Therefore all the f_i , $i=1, \dots, n$, in (3.5) are s.i.p. To finish the proof, take $t=1$ in (3.3).

COROLLARY B. Among the irreducible s.i.p. there are certainly two polynomials with independent linear terms.

Proof. Apply Proposition 1 and Corollary A to $x, y \in k[x, y]$.

4. Invariant functions. Among the rational functions on A^2 some are invariant under σ . In the next few lemmas we investigate the connexions between those functions and the s.i.p.

LEMMA 3. *The field of rational functions on A^2 which are invariant under σ is a simple extension of k .*

Proof. Consider the variety W of orbits (see [3]) corresponding to the operation of G_m on A^2 . W is of dimension one. Its function-field $k(W)$ is by Lüroth's theorem (see [4]) a simple extension of k . On the other hand $k(W)$ is exactly the subfield of $k(x, y)$ containing those functions of $k(x, y)$ which are invariant under σ . Q.E.D.

If σ is such that all s.i.p. are of positive weight we have $k(W) = k(f/g)$, where f and g are s.i.p. of the same weight (see [2, p. 84]). In the sequel $k(W)$ will denote the subfield of $k(x, y)$ formed by the functions on A^2 invariant under σ .

LEMMA 4. *Let σ be such that all s.i.p. are of positive weight and let f/g , (f and g relatively prime) be a generator for $k(W)$. Consider an element $p = f + \mu g$, $\mu \in k$. Then all the irreducible factors of p are equal.*

Proof. If f/g is a generator for $k(W)$, then so is $f + \mu g/g$. Therefore it suffices to prove that all the irreducible factors of f are equal. Suppose that $f = f_1^\alpha f_2$, where f_1 is irreducible and f_2 contains no factors equal to f_1 . Since the weights of f, g, f_1 are all positive $f_1^\alpha g^{-b}$ will be an invariant function for some well chosen a and b . Then $f_1^\alpha g^{-b} \in k(f_1^\alpha f_2 g^{-1})$ i.e., since k is algebraically closed

$$\frac{f_1^\alpha}{g^b} = \frac{c \prod_{i=1}^n ((f_1^\alpha f_2/g) + a_i)}{\prod_{j=1}^m ((f_1^\alpha f_2/g) + b_j)} \text{ with } c, a_i, b_j \in k,$$

or

$$(4.1) \quad f_1^\alpha \prod_{j=1}^m (f_1^\alpha f_2 + b_j g) = c g^\varepsilon \prod_{i=1}^n (f_1^\alpha f_2 + a_i g),$$

(if ε is negative, we make the obvious change needed in order to get a polynomial identity).

Now f_1^α divides the right-hand member of (4.1). Since $f_1 \nmid g$ and $f_1 \nmid f_2$, it follows by standard divisibility arguments that $f_2 = 1$.

LEMMA 5. *Let a regular operation of G_m on A^2 be such that all s.i.p. are of positive weight. Let p_1, p_2 and p_3 be three irreducible s.i.p. of weights ω_1, ω_2 and ω_3 respectively, such that $p_1, p_2, p_3 \notin k$.*

Then they satisfy a relation of the type

$$(4.2) \quad c_1 p_1^\alpha + c_2 p_2^\beta = p_3^\gamma$$

where $\alpha\omega_1 = \beta\omega_2$, $(\alpha, \beta) = 1$; $c_1, c_2 \in k$.

Proof. Let d be the g.c.d. of ω_1 and ω_2 . Then $\omega_1 = \beta d$ and $\omega_2 = \alpha d$. Consider $\phi = p_1^\alpha p_2^{(-\beta)}$. ϕ is invariant under σ , so that, if fg^{-1} is a generator of $k(W)$ we have $p_1^\alpha p_2^{(-\beta)} \in k(fg^{-1})$. Therefore

$$p_1^\alpha p_2^{(-\beta)} = \frac{c \prod_{i=1}^m (f/g - a_i)}{\prod_{j=1}^n (f/g - b_j)} \text{ with } c, a_i, b_j \in k,$$

or

$$(4.3) \quad p_1^\alpha \prod_{j=1}^n (f - b_j g) = c p_2^\beta g^\varepsilon \prod_{i=1}^m (f - a_i g),$$

(if ε is negative, we multiply both members of (4.3) by $g^{-\varepsilon}$, in order to get a polynomial identity).

Using Lemma 4 we then have, by standard divisibility arguments,

$$(4.4) \quad p_1^\alpha = d_1 f + d_2 g, \quad p_2^\beta = d_3 f + d_4 g, \quad d_1, d_2, d_3, d_4 \in k,$$

whence we can choose as generator for $k(W)$ the element $p_1^\alpha p_2^{(-\beta)}$. We now repeat this argument with the invariant function $p_2^{\omega_3} p_3^{(-\omega_2)}$, taking $p_1^\alpha p_2^{(-\beta)}$ as a generator for $k(W)$. The relation which we then get instead of (4.4) is

$$p_3^\gamma = c_1 p_1^\alpha + c_2 p_2^\beta, \quad c_1, c_2 \in k,$$

where γ is another positive number.

5. Invariant polynomials. Among the functions invariant under σ there may, or may not, be polynomials. The two following lemmas treat both possibilities, and lead up to the main theorem.

LEMMA 6. *If there exist nonconstant polynomials invariant under σ there is at most one irreducible s.i.p. of positive weight and similarly at most one irreducible s.i.p. of negative weight.*

Proof. Let f be an invariant polynomial of the lowest degree, $f \notin k$, and let $k(W) = k(g/h)$, where g and h are relatively prime polynomials. Then

$$f = \frac{c \prod_{i=1}^n (g/h - a_i)}{\prod_{j=1}^m (g/h - b_j)} = \frac{ch^\varepsilon \prod_{i=1}^n (g - a_i h)}{\prod_{j=1}^m (g - b_j h)} \text{ with } c, a_i, b_j \in k$$

for some suitable ε . It follows that some $g - b_j h$ is a constant, and therefore that $k(W) = k(h) = k(f)$.

Suppose now that p_1 and p_2 are different irreducible s.i.p. of positive weight. Then for an appropriate choice of a and b , $a > 0$, $b > 0$, $\phi = p_1^a p_2^{-b}$ is invariant under σ . Therefore $p_1^a p_2^{-b} \in k(f)$. We then get a polynomial identity

$$p_1^a \prod_{j=1}^m (f - c_j) = d p_2^b \prod_{i=1}^m (f - d_i) \quad \text{with } d, c_j, d_i \in k.$$

As p_1 and p_2 are both irreducible it then follows that $p_1^a \in k(f)$, $p_2^{-b} \in k(f)$, which is a contradiction. The proof is similar for p_1 and p_2 both of negative weight.

LEMMA 7. *Let σ be such that there are no nonconstant invariant polynomials. Then after a suitable change of coordinates (a translation) all s.i.p. will be polynomials without constant terms.*

Proof. If there are no invariant polynomials the s.i.p. are either all of positive or all of negative weight. It is no loss of generality to take all weights positive.

Let $\sigma(t)$ be given by

$$\begin{aligned} x^* &= t^a [f_0 + t f_1 + \cdots + t^n f_n], \\ y^* &= t^b [g_0 + t g_1 + \cdots + t^m g_m]. \end{aligned} \tag{5.1}$$

By (3.6) f_i is an s.i.p. of weight $(\alpha + i)$. Since, by assumption, all s.i.p. are of positive weight we may suppose $\alpha = \beta = 0$ in (5.1). On the other hand f_0 and g_0 are then reduced to constants, since there are no invariant polynomials. Hence (5.1) becomes

$$\begin{aligned} x^* &= a + t f_1(x, y) + \cdots + t^n f_n(x, y), \\ y^* &= b + t g_1(x, y) + \cdots + t^m g_m(x, y), \end{aligned} \tag{5.2}$$

where $m > 0$, $n > 0$, $a, b \in k$.

After the change of coordinates

$$X = x - a, \quad Y = y - b, \tag{5.3}$$

we get

$$\begin{aligned} X^* &= t f_1^*(X, Y) + \cdots + t^n f_n^*(X, Y), \\ Y^* &= t g_1^*(X, Y) + \cdots + t^m g_m^*(X, Y), \end{aligned} \tag{5.4}$$

where $f_i^*(X, Y) = f_i(X + a, Y + b)$, $g_j^*(X, Y) = g_j(X + a, Y + b)$. In this new system of coordinates no s.i.p. of positive weight can have a constant term, since all powers of t in the right-hand members of (5.4) are positive.

PROPOSITION 2. *Let σ be a regular operation of G_m on A^2 . Then the set of all polynomials of $k[x, y]$ semi-invariant under σ is generated over k by two of its elements.*

Proof. We distinguish two cases:

(A) *There are no invariant polynomials.* Then all weights may be taken positive. By Lemma 7 we may suppose that all s.i.p. have no constant terms. We know (by Corollary B) that among the s.i.p. there are at least two p_1 and p_2 , with independent linear terms. Since all s.i.p. have no constant term, p_1 and p_2 are irreducible. Denote by ω_1 , and ω_2 the weights of p_1 and p_2 respectively. Let p_3 of weight ω_3 be any other irreducible s.i.p. By Lemma 5 there exists a relation of the type:

$$(5.5) \quad p_3^\gamma = c_1 p_1^\alpha + c_2 p_2^\beta, \quad c_1, c_2 \in k,$$

where $\alpha d = \omega_2$, $\beta d = \omega_1$, $(\alpha, \beta) = 1$. Then $\gamma \omega_3 = \alpha \beta d$. Let $\alpha \leq \beta$. The polynomials p_1 and p_2 have independent linear terms—so that p_3^γ will have among its terms of lowest degree a term of the type $c_3 x^\delta y^\varepsilon$ with $\delta + \varepsilon = \alpha$, $c_3 \in k$. Therefore $\alpha = \mu \gamma$. Hence $\omega_3 = \mu \beta d = \mu \omega_1$. Applying Lemma 5 again, this time to p_3 , p_1 and p_2 , we get

$$(5.6) \quad c_5 p_3 + c_6 p_1^\mu = p_2^\varepsilon, \quad c_5, c_6 \in k.$$

So that p_3 is a polynomial in p_1 and p_2 .

(B) *There exist invariant polynomials.* By Lemma 6 there are at most two irreducible s.i.p., one of negative and one of positive weight. To conclude the proof it remains to show that they generate over k all the invariant polynomials. Let p_1 and p_2 be these two irreducible polynomials of weights ω_1 and ω_2 respectively, $\omega_1 > 0$, $\omega_2 < 0$. Let f be a generator for $k(W)$. Then $p_1^{(-\omega_2)} p_2^{\omega_1} \in k(f)$. The desired result follows by a divisibility argument, using the same methods as in Lemmas 4 and 5.

6. Main theorems.

PROPOSITION 3. *Any regular operation of G_m on A^2 can, after a suitable change of coordinates, be reduced to the form*

$$(6.1) \quad (t, x, y) \rightarrow (t^\mu x, t^\nu y)$$

with μ and ν integers.

Proof. Let σ be given by $(t, x, y) \rightarrow (x^*, y^*)$ where

$$(6.2) \quad x^* = t^\alpha \sum_{i=0}^n f_i t^i,$$

$$y^* = t^\beta \sum_{j=0}^m g_j t^j,$$

with $m \geq 0$, $n \geq 0$, $f_i, g_j \in k[x, y]$.

By Proposition 1 the f_i and g_j are s.i.p. By Proposition 2 all s.i.p. are polynomials in two irreducible s.i.p., say X and Y . Then $f_i, g_j \in k[X, Y]$ and $X, Y \in k[x, y]$.

On the other hand $x = \sum f_i$, $y = \sum g_j$, so that x and y are polynomials in X and Y , say $x = P(X, Y)$, $y = Q(X, Y)$. Therefore we have the relations

$$(6.3) \quad \begin{cases} x = P(X, Y), \\ y = Q(X, Y), \end{cases} \quad \begin{cases} X = X(x, y), \\ Y = Y(x, y), \end{cases}$$

which define an allowable change of coordinates in A^2 . In terms of the (X, Y) -coordinate system σ is then given by

$$(6.4) \quad (t, X, Y) \rightarrow (t^\mu X, t^\nu Y)$$

since X and Y are s.i.p.

PROPOSITION 4. *If G_m operates regularly on A^2 there is always a fixed point.*

Proof. This is an obvious corollary of the previous proposition. Note that G_m can operate on A^2 in such a way that there exists a curve of fixed points, but that in this case there is a fixed point adherent to every orbit.

7. Algebraic torus. The next two propositions concern the action of an algebraic torus on A^2 .

PROPOSITION 5. *If an algebraic torus operates regularly on A^2 there always exists a fixed point.*

Proof. Consider the algebraic torus $G_m^{(1)} \times G_m^{(2)}$ and let σ_1 and σ_2 respectively be the regular operations of $G_m^{(1)}$ and $G_m^{(2)}$ on A^2 . Since $G_m^{(1)}$ and $G_m^{(2)}$ commute, the set of orbits corresponding to σ_1 is globally invariant under σ_2 , and so is the set of fixed points of σ_1 . Therefore, if σ_i has only one fixed point, it is also a fixed point for σ_j , $i \neq j$. If on the other hand both σ_1 and σ_2 have curves of fixed points, the curve of fixed points of σ_1 is either a curve of fixed points for σ_2 , or an orbit of σ_2 . In either case it has at least one fixed point. This proves that σ_1 and σ_2 have a common fixed point. The proposition then follows by induction.

PROPOSITION 6. *Let an algebraic torus $G_m^{(1)} \times G_m^{(2)} \times \cdots \times G_m^{(r)}$ operate regularly on A^2 . Then after an appropriate change of coordinates this operation can be described by*

$$(7.1) \quad (s_1, s_2, \dots, s_r, x, y) \rightarrow \left(x \prod_{i=1}^r s_i^{\alpha_i}, y \prod_{j=1}^r s_j^{\beta_j} \right),$$

where $s_i \in G_m^{(i)}$, $i = 1, 2, \dots, r$, and the α_i and β_j are integers.

Proof. Let a regular operation of $G_m \times G_m$ on A^2 be given by

$$(7.2) \quad (s, t, x, y) \rightarrow (x^*, y^*)$$

with $x^*, y^* \in k[x, y, s, s^{-1}, t, t^{-1}]$.

In particular (7.2) defines by restriction two regular operations of G_m on A^2 which we denote by

$$(7.3) \quad \sigma_1: (s, x, y) \rightarrow (x_1, y_1),$$

$$(7.4) \quad \sigma_2: (t, x, y) \rightarrow (x_2, y_2).$$

Suppose that the coordinate system is chosen so that σ_2 is of the form

$$(7.5) \quad \sigma_2: (t, x, y) \rightarrow (t^\alpha x, t^\beta y).$$

Obviously in some allowable coordinate system σ_1 is of the form

$$(7.6) \quad (s, x', y') \rightarrow (s^\gamma x', s^\delta y')$$

and in this coordinate system the orbits have as equation

$$(7.7) \quad x'^\delta + \mu y'^\gamma = 0$$

if we suppose that $\gamma\delta > 0$. Let the change of coordinates from (x', y') to (x, y) be given by

$$(7.8) \quad x' = f(x, y), \quad y' = g(x, y), \quad f, g \in k[x, y].$$

Then in the (x, y) -coordinate system the equation of the orbits (7.7) is

$$(7.9) \quad f^\delta + \mu g^\gamma = 0.$$

These orbits have to be globally invariant under σ_2 , so that f^δ/g^γ is a semi-invariant function for σ_2 . By Lemma 2, f^δ and g^γ are then s.i.p., which means that both f and g are s.i.p. for σ_2 , that is, in the (x', y') -coordinate system x' and y' are s.i.p. for σ_2 . This proves that in the (x', y') -coordinate system both σ_1 and σ_2 are reduced to the canonical form of Proposition 3. The proof is similar if $\gamma\delta \leq 0$. Our proposition follows by induction.

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